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LAGRANGIAN FIBRATIONS ON MODULI SPACES OF SINGULAR CONNECTIONS ON CURVES

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ABSTRACT. In this note, we will report some recent works on the geometry of moduli spaces of singular connections on smooth projective curves. These works are based on the joint works with M. Inaba [9], S. Szabo [15], F.Loray-C. Simpson ([11], [12])).

1. The Construction of Moduli spaces of stable parabolic connections

Fix integers $g \geq 0, n > 0, r > 0, d$ and let C be a nonsingular projective curve of genus g and $\mathbf{t} = \{t_1, \cdots, t_n\}$ a set of ordered n-distinct points on C. For a data $\mathbf{t} = \{t_1, \cdots, t_n\}$, we denote by $D = D(\mathbf{t}) = t_1 + \cdots + t_n$ the divisor associated to $\mathbf{t}$. We will review known results on the moduli space of stable parabolic connections in [5], [6] and [7].

A logarithmic connection on C with singularities at D is a pair (E, ∇) where E is an algebraic (or holomorphic) vector bundle on C of rank r and degree d and a morphism of sheaves $\nabla : E \longrightarrow E \otimes \Omega^1_C(D)$ which satisfies the Leibniz rule, i.e., for local sections $a \in \mathcal{O}_C, \sigma \in E$, $\nabla(a\sigma) = \sigma \otimes da + a\nabla(\sigma)$.

For such a logarithmic connection (E, ∇), we can define a residue homomorphism $\mathsf{res}_{t_i}(\nabla) \in \mathrm{End}(E_{|t_i}) \simeq M_r(\mathbf{C})$ for each $i, 1 \leq i \leq n$. Let $\{\nu_0^{(i)}, \nu_1^{(i)}, \cdots, \nu_{r-1}^{(i)}\}$ be the set of (ordered) eigenvalues of $\mathsf{res}_{t_i}(\nabla)$ which are called *local exponents* of ∇ at t_i . Moreover we define the set of all local exponents $\boldsymbol{\nu}$ of (E, ∇) by $\boldsymbol{\nu} = (\nu_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1}$.

The following lemma is a *generalization of Fuchs relation* when $C \simeq \mathbf{P}^1$ and can be proved by the residue formula.

Lemma 1.1. *For a logarithmic connection (E, ∇) with singularity at $D = D(\mathbf{t})$ as above, let $\boldsymbol{\nu} = (\nu_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1}$ be the set of local exponents of (E, ∇). Then one has*

$$\sum_{i=1}^{n}(\sum_{j=0}^{r-1} \nu_j^{(i)}) = -\deg E = -\deg \wedge^r E = -d. \tag{1}$$

For each (n, r, d), we define the set of local exponents by

$$\mathcal{N}_r^n(d) := \left\{ \boldsymbol{\nu} = (\nu_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1} \in \mathbf{C}^{nr} \;\middle|\; d + \sum_{1 \leq i \leq n} \left(\sum_{0 \leq j \leq r-1} \nu_j^{(i)} \right) = 0 \right\} \simeq \mathbf{C}^{nr-1} \tag{2}$$

Definition 1.1. For $(C, \mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$, a *$\boldsymbol{\nu}$-parabolic connection* on C with singularities at $D = D(\mathbf{t})$ is a collection $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ consisting of the following data:

(1) a logarithmic connection (E, ∇) on C with singularities at D.

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(2) and a filtration $l_*^{(i)} : E_{|t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ for each $i, 1 \leq i \leq n$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\mathrm{res}_{t_i}(\nabla) - \nu_j^{(i)} Id)(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, 1, \cdots, r-1$.

For $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$, we set $r = \operatorname{rank} E$, $d = \deg E$.

Note that for each fixed i, $1 \leq i \leq n$, $\{\nu_j^{(i)}\}_{0 \leq j \leq r-1}$ is the set of ordered eigenvalues of the residue matrix $\mathrm{res}_{t_i}(\nabla)$, so the parabolic structure $\{l_*^{(i)}\}$ or its successive quotients $\{l_j^{(i)}/l_{j+1}^{(i)}\}$ give the eigenspaces for $\mathrm{res}_{t_i}(\nabla)$.

Since we will construct moduli spaces of $\boldsymbol{\nu}$-parabolic connections as smooth quasi-projective schemes, it is natural to introduce the stability condition on the $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$.

A weight $\boldsymbol{\alpha} = \{\alpha_j^{(i)}\}_{1 \leq j \leq r}^{1 \leq i \leq n}$ is a sequence of rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1 \tag{3}$$

for $i = 1, \ldots, n$. (Later, we will assume some generic conditions like $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i,j) \neq (i', j')$).

Let $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ be a $\boldsymbol{\nu}$-parabolic connection, and $F \subset E$ a nonzero subbundle satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D)$. We define integers $\mathrm{length}(F)_j^{(i)}$ by

$$\mathrm{length}(F)_j^{(i)} = \dim(F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}). \tag{4}$$

Note that $\mathrm{length}(E)_j^{(i)} = \dim(l_{j-1}^{(i)}/l_j^{(i)}) = 1$ for $1 \leq j \leq r$.

Definition 1.2. A $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is *$\boldsymbol{\alpha}$-stable* if for any proper nonzero subbundle $F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D(\mathbf{t}))$, the inequality

$$\frac{\deg F + \sum_{i=1}^m \sum_{j=1}^r \alpha_j^{(i)} \mathrm{length}(F)_j^{(i)}}{\operatorname{rank} F} < \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \mathrm{length}(E)_j^{(i)}}{\operatorname{rank} E} \tag{5}$$

holds.

For a fixed $(C, \mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$, let us define the coarse moduli space by

$$\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, d) = \left\{ (E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n}) \ \middle| \ \begin{array}{l} \text{an } \boldsymbol{\alpha}\text{-stable } \boldsymbol{\nu}\text{-parabolic connection} \\ \text{of rank } r \text{ and degree } d \text{ over } C \end{array} \right\} / \simeq . \tag{6}$$

Here the equivalence relation $\simeq$ is given by natural isomorphisms between $\boldsymbol{\nu}$-parabolic connections.

Let $M_{g,n}$ be the moduli space of n-pointed smooth projective curves $(C, \mathbf{t})$. Taking a finite covering $\tilde{M}_{g,n} \longrightarrow M_{g,n}$, we may assume that there exists the universal family $(\mathcal{C}, \tilde{\mathbf{t}}) = (\mathcal{C}, \tilde{t}_1, \cdots, \tilde{t}_n)$ over $\tilde{M}_{g,n}$. We have the following fundamental result for the construction of the moduli space (cf. [5], [6]).

Theorem 1.1. *For sufficiently generic weight $\boldsymbol{\alpha}$, there exists a relative fine moduli scheme*

$$\pi : \mathcal{M}_{(\mathcal{C},\tilde{\mathbf{t}})/\tilde{M}_{g,n} \times \mathcal{N}_r^n(d)}^{\boldsymbol{\alpha}}(r, d) \longrightarrow \tilde{M}_{g,n} \times \mathcal{N}_r^n(d) \tag{7}$$

of $\boldsymbol{\alpha}$-stable parabolic connection of rank r and degree d. The morphism π is smooth and quasi-projective. The fiber of π over $((C, \mathbf{t}), \boldsymbol{\nu}) \in \tilde{M}_{g,n} \times \mathcal{N}_r^n(d)$ is isomorphic to the moduli space $\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, d)$ in (6). Hence the moduli space $\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, d)$ is a smooth quasi-projective algebraic scheme, whose dimension is $2r^2(g-1) + nr(r-1) + 2$ if it is non-empty.

Remark 1.1. In the regular singular case, Inaba [5] shows rigorously the geometric Painlevé property of differential equations arising from the isomonodromic deformations of singular connections on curves. (See also [6] and [7]). We also established the existence of the moduli spaces of irregular singular parabolic connections with irregular singularities of generic unramified fixed formal types as a quasiprojective smooth scheme. ([9]). Moreover, we can show that the generalized Riemann-Hilbert correspondences are isomorphisms for generic formal types. This result give a rigorous proof of the fact that the geometric Painlevé property of the differential equations arising from isomonodromic and iso-Stokes deformations of irregular connections on curves with generic formal types. Note that geometric Painlevé property implies the usual analytic Painlevé property.

2. Deformation theory and Symplectic structure

2.1. **Infinitesimal deformations.** We will describe the tangent space of $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ at a point $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n})$ by the infinitesimal deformation theory of parabolic connections. A canonical holomorphic symplectic structure on $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ can be defined by the non-degenerate natural pairing on the hyper-cohomology group induced by the Serre duality.

For $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$, we can define a complex of sheaves $\mathcal{F}^\bullet$

$$
\begin{aligned}
&(8) & \mathcal{F}^0 &:= \mathcal{E}nd(E)_p = \{s \in \mathcal{E}nd(E) | s_{|t_i}(l^{(i)}_j) \subset l^{(i)}_j \text{for any } i, j\} \\
&(9) & \mathcal{F}^1 &:= (\mathcal{E}nd(E) \otimes \Omega^1_C(D))_{isom} \\
& & &= \{s \in \mathcal{E}nd(E) \otimes \Omega^1_C(D) | \operatorname{res}_{t_i}(s)(l^{(i)}_j) \subset l^{(i)}_{j+1} \text{for any } i, j\ \}
\end{aligned}
$$

$$
\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \longrightarrow \mathcal{F}^1, \quad \nabla_{\mathcal{F}^\bullet}(s) = \nabla \circ s - s \circ \nabla.
$$

We have the following proposition (cf. ([1], [5], [§6.1, [6]]).

Proposition 2.1. *The infinitesimal deformation of $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ can be described by hypercohomlogy groups $\mathbf{H}^\bullet(\mathcal{F}^\bullet) = \mathbf{H}^\bullet(\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \longrightarrow \mathcal{F}^1)$. The space of infinitesimal deformations of $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n})$ is isomorphic to $\mathbf{H}^1(\mathcal{F}^\bullet)$ and the set of obstructions of infinitesimal deformations are contained in $\mathbf{H}^2(\mathcal{F}^\bullet)$. Moreover since $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n})$ is $\boldsymbol{\alpha}$-stable, all obstructions vanish, hence the moduli space $\mathcal{M} = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ is smooth everywhere, and the tangent space at the point $\mathbf{x} = (E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n})$ is isomorphic to the hypercohomology group $\mathbf{H}^1(\mathcal{F}^\bullet)$, that is,*

$$
T_{\mathcal{M},\mathbf{x}} \simeq \mathbf{H}^1(\mathcal{F}^\bullet).
$$

2.2. **Symplectic structure.** From the spectral sequence associated to the hypercohomology group, we have the following exact sequence

$$
0 \to \mathbf{C} \to H^0(\mathcal{F}^0) \to H^0(\mathcal{F}^1) \to \mathbf{H}^1(\mathcal{F}^\bullet) \to H^1(\mathcal{F}^0) \to H^1(\mathcal{F}^1) \to \mathbf{C} \to 0. \tag{10}
$$

The following lemma is a key.

Lemma 2.1. *There exists an isomorphism*

$$
(\mathcal{F}^0)^\vee \otimes \Omega^1_C \simeq \mathcal{F}^1.
$$

Hence, together with Serre duality, we have following natural isomorphisms

$$\begin{aligned} H^0(C,\mathcal{F}^0) &\simeq H^1(C,(\mathcal{F}^0)^\vee\otimes\Omega^1)^\vee\simeq H^1(C,\mathcal{F}^1)^\vee \\ H^0(C,\mathcal{F}^1) &\simeq H^0(C,(\mathcal{F}^0)^\vee\otimes\Omega^1_C)\simeq H^1(C,\mathcal{F}^0)^\vee \end{aligned}$$

Theorem 2.1. *The exact sequence (10) is self-dual, hence,*

$$\mathbf{H}^1(\mathcal{F}^\bullet)\simeq\mathbf{H}^1(\mathcal{F}^\bullet)^\vee.$$

In particular, the tangent space of the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$ at each point $\mathbf{x}$ has a natural non-degenerate pairings.

Remark 2.1. Theorem 2.1 implies that there exists a non-degenerate 2-form ω on the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$, which gives a symplectic structure on the moduli space. Inaba [5] showed that the non-degenerate 2-form ω is d-closed. Note that the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$ is never projective if its dimension is positive.

3. Moduli stack of quasi-parabolic bundles and forgetful Lagrangian fibrations

Let $\widehat{\mathcal{P}}^r_d$ be the moduli stack of quasi-parabolic bundles $(E,l)=(E,\{l^{(i)}_*\}_{1\le i\le n})$ of rank r and degree d over $(C,\mathbf{t})$. We say that a quasi-parabolic bundle (E,l) is simple if $H^0(C,\mathcal{F}^0)=\mathbf{C}$, that is, every endomorphism of E preserving the quasi-parabolic structure $l=\{l^{(i)}_*\}_{1\le i\le n}$ is just a scalar multiplication. We denote by $\widehat{\mathcal{P}}^{r,simple}_d\subset\widehat{\mathcal{P}}^r_d$ the moduli substack of simple quasi-parabolic bundles.

Let us also denote by $\widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$ the moduli stack of $\boldsymbol{\alpha}$-stable connection (E,∇,l) with the given invariants. For a generic weight $\boldsymbol{\alpha}$, this stack has the fine moduli space as we see in Theorem 1.1:

$$\widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)\longrightarrow\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d).$$

We have a natural forgetful morphism of stacks

$$\pi:\widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)\longrightarrow\widehat{\mathcal{P}}^r_d,\quad \pi((E,\nabla,\{l^{(i)}_*\}_{1\le i\le n}))=(E,\{l^{(i)}_*\}_{1\le i\le n}).$$

Let $\overline{\mathcal{P}^r_d}$ be the image of π. In general, it seems difficult to characterize the image $\overline{\mathcal{P}^r_d}$. When $C=\mathbf{P}^1$ and $r=2$, assuming that the local exponent $\boldsymbol{\nu}\in\mathcal{N}^n_r(d)$ is generic ([2]), Arinkin and Lysenko showed that $\overline{\mathcal{P}^2_0}$ coincides with the substack $\widehat{\mathcal{P}}^{2,simple}_0$ corresponding to simple quasi-parabolic bundles.

Under the assumption that $\boldsymbol{\nu}$ is generic, we see that every $\boldsymbol{\nu}$-parabolic connection (E,∇,l) is irreducible, hence $\boldsymbol{\alpha}$-stable for any choice of $\boldsymbol{\alpha}$. Assuming this, for arbitrary genus and rank, as in [Proposition 3, [2]], we can show that a simple quasi-parabolic bundle (E,l) is in the image $\overline{\mathcal{P}^r_d}$, and hence $\widehat{\mathcal{P}}^{r,simple}_d$ is an open substack of $\overline{\mathcal{P}^r_d}$.

Let us define the substack $\widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)^0\subset\widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$ which corresponds to the inverse image $\pi^{-1}(\widehat{\mathcal{P}}^{r,simple}_d)$. Moreover we denote by the corresponding open subscheme by $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)^0$ of the fine moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$.

For simplicity, we assume the following

Assumption 3.1. *The substack $\widehat{\mathcal{P}}_d^{r,simple}$ has a coarse moduli space $\mathcal{P}_d^r$*

$$\widehat{\mathcal{P}}_d^{r,simple} \longrightarrow \mathcal{P}_d^r.$$

It is known that the coarse moduli space $\mathcal{P}_d^r$ exists as an algebraic space.

Note that as in [2], $\mathcal{P}_d^r$ may be a non-separated scheme. See also Section 6 in [11].

Under the assumption, we have the following commutative diagram.

$$\begin{array}{ccc} \pi : \widehat{\mathcal{M}}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)^0 & \longrightarrow & \widehat{\mathcal{P}}_d^{r,simple} \\ \downarrow & & \downarrow \\ \pi_1 : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)^0 & \longrightarrow & \mathcal{P}_d^r \end{array}$$

Assume that on a Zariski open set U of $\mathcal{P}_d^r$ there exists a section (E, ∇_0, l) over $\mathbf{x} = (E, l) \in U$. Then if we take a point $(E, \nabla, l) \in \pi_1^{-1}(U)$, we see that $\nabla - \nabla_0 = \omega_\nabla \in H^0(C, \mathcal{F}^1) \simeq T^*_{\mathbf{x}}$, hence $\nabla = \nabla_0 + \omega_\nabla$. This implies that $\pi_1^{-1}(U) \longrightarrow U$ is isomorphic to the cotangent bundle $T^*_U \longrightarrow U$ with 0-section ∇_0. Hence we can show the following

Theorem 3.1. *Under the assumption, the map π_1 gives a $T^*_{\mathcal{P}_d^r}$-torsor structure on $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)^0$. Hence every fiber of π_1 over a closed point of $\mathcal{P}_d^r$ is an affine space space $H^0(C, \mathcal{F}^1) \simeq \mathbf{C}^N$, which is Lagrangian subvariety with respect to the natural symplectic structure. We call the map π_2 a forgetful Lagrangian fibration.*

Remark 3.1. In the case of $C = \mathbf{P}^1$ and $r = 2$, assume that $\boldsymbol{\nu}$ is generic so that every $\boldsymbol{\nu}$-parabolic connection is irreducible. In this case $\mathcal{P}_0^2$ is a smooth non-separated irreducible scheme of dimension $n - 3$ ([2]) and the tangent space $T_{\mathcal{P}_0^2, \mathbf{x}}$ at each $\mathbf{x} = (E, l)$ is $H^1(\mathbf{P}^1, \mathcal{F}^0)$. Moreover $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)^0 = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ and hence we have a surjective smooth morphism

$$\pi_1 : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1,\mathbf{t})}(\boldsymbol{\nu}, 2, 0) \longrightarrow \mathcal{P}_0^2.$$

We describe this morphism π_1 in detail for $n = 4$ later. (See also Section 9, [11]).

4. Canonical coordinates of moduli spaces of connections from apparent singularities and the associated Lagrangian fibrations

4.1. **Apparent singularities of connections.** Let $(E, \nabla, l) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ be an irreducible $\boldsymbol{\nu}$-parabolic connection of rank r and of degree d. In this section, we always assume that $n \geq 1$. In this case, we can change $d = \deg E$ freely by elementary transformations at singular points $t_i \in C$. Assume that $\dim H^0(C, E) > 0$ and take a non-zero section $\sigma \in H^0(C, E)$. We can introduce apparent singularities of (E, ∇, l) by using this section σ as a cyclic vector of ∇. It is classical to use the apparent singularities and its symplectic dual coordinates as a canonical coordinate system of the moduli space of the connections ([3], [10], [14]). Moreover the isomonodromic deformation of the linear connections can be written in this coordinate system.

In [15], we introduce a systematic way to introduce apparent singularities which in turn coincide with the classical ones on a Zariski open set of the moduli space.

Let $(E, \nabla, \{l_*^{(i)}\})$ be a parabolic connection with logarithmic singularities at $D = D(\mathbf{t})$ such that $\operatorname{rank} E = r$, $\deg E = d$. Let us set $L = \Omega_C^1(D)$ and note that $\deg L = 2g - 2 + n$.

Proposition 4.1. *Assume that there exists a non-zero section $\sigma \in H^0(C,E)$ and $\deg L = 2g-2+n \geq 1$ and $\deg D = n \geq 1$. Moreover assume that (E,∇) is an irreducible connection. Consider the direct sum of line bundles*

$$F = \oplus_{j=0}^{r-1} L^{-j} = \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(r-1)}. \tag{11}$$

Then we have a natural embedding $F \hookrightarrow E$ such that $H^0(C,F) \simeq \mathbf{C}\sigma \subset H^0(C,E)$. Let us define the torsion sheaf T_A by the exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0, \tag{12}$$

Then torsion sheaf T_A has the length

$$N = d - r(g-1) + r^2(g-1) + n\frac{r(r-1)}{2}.$$

We note that F is a parabolic version of oper which is successive extension of $\mathcal{O}_C$ by $L^{-j} = (\Omega^1_C(D))^{-j}$. If $\deg D > 0$, all extension classes vanish, hence F is the direct sum of line bundles as in (11). Since $\deg L$ is positive, $H^0(C,F) \simeq H^0(C,\mathcal{O}_C) \simeq \mathbf{C}\sigma$. Then applying ∇^j to this section σ, we have successive extension of $\mathcal{O}_C$ by L^{-j}. See [15] for detail.

Definition 4.1. *For an irreducible parabolic connection (E,∇,l) and a non-zero section σ, we call the support of T_A* apparent singular points *of the parabolic connection (E,∇,l) with* the cyclic vector *σ.*

Now assume that $\deg E = d = r(g-1)+1$. We have $\dim H^0(C,E) = \dim H^1(C,E) + 1$ by Riemann-Roch. If moreover $H^1(C,E) = 0$, we have a non-zero section $\sigma \in H^0(C,E) \simeq \mathbf{C}\sigma$ unique up to non-zero scalar multiplications.

Corollary 4.1. *Under the same notation and assumption as in Proposition 4.1, let us assume that*

$$d = \deg E = r(g-1)+1, \tag{13}$$

$$H^1(C,E) = 0. \tag{14}$$

Then we have a natural unique embedding $F \hookrightarrow E$ which yields

$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0, \tag{15}$$

Then the sheaf T_A is a torsion sheaf of length

$$N = r^2(g-1) + n\frac{r(r-1)}{2} + 1. \tag{16}$$

4.2. **Generic case.** Now we consider the case of $d = \deg E = r(g-1)+1$. We assume that there exists an irreducible connection $\boldsymbol{\nu}$-parabolic connection $(E,\nabla,l) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$ with $\dim H^1(C,E) = 0$. Then we have a non-empty Zariski open $\mathcal{M}^0 \subset \mathcal{M}^{\boldsymbol{\alpha}}_{C,\mathbf{t}}(\boldsymbol{\nu},r,d)$ whose points correspond to irreducible connections with the condition $\dim H^1(C,E) = 0$. For such a connection $(E,\nabla,l) \in \mathcal{M}^0$, from Corollary 4.1, we have a unique canonical exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0, \quad F = \oplus_{i=1}^{r} L^{-(j-1)}. \tag{17}$$

Assume also that the support of T_A consists of N-distinct apparent singular points (which corresponds to a Zariski open set $\mathcal{M}^{00} \subset \mathcal{M}^0$)

$$q_1, \cdots, q_N \in C$$

where $N = r^2(g-1) + n\frac{r(r-1)}{2} + 1$. (Here we put the numbering of apparent singular points.) Then we have the decomposition

$$T_A \simeq \oplus_{i=1}^N \mathbf{C}_{q_i}.$$

Fix a connection $\nabla_0 : E \longrightarrow E \otimes \Omega_C^1(D) = E \otimes L$ such that $\nabla_0 = d$ on $E_{|C \setminus \{t_n\}}$ and the eigenvalues of $\mathrm{res}_{t_n}(\nabla_0)$ are integers (cf. [15]). Then for the connection ∇ above, we set $\Phi(\nabla) := \nabla - \nabla_0 : E \longrightarrow E \otimes L$. We see that $\Phi(\nabla)$ is $\mathcal{O}_C$-linear and hence $\Phi(\nabla) \in H^0(C, \mathrm{End}(E) \otimes L)$. Then we have a morphism $\Phi(\nabla) : T_A \longrightarrow T_A \otimes L$ which decompose into $\mathbf{C}_{q_i}$-linear morphisms $\Phi(\nabla)_{q_i} : \mathbf{C}_{q_i} \longrightarrow \mathbf{C}_{q_i} \otimes L_{q_i}$.

Definition 4.2. We denote $p_i \in L_{q_i}$ as the scalar corresponding to $\Phi(\nabla)_{q_i}$. We call the set of apparent singular points $\{q_1, \cdots, q_N\}$ with their duals$\{p_1, \cdots, p_N\}$ the canonical coordinate system for the Zariski open set $\mathcal{M}^{00}$ of the moduli space of $\boldsymbol{\nu}$-parabolic connections.

We will make this definition more precise. Let $L \longrightarrow C$ be the total space of line bundle L. For each $i, 1 \leq i \leq n$, we can find r-points $\{b_j^{(i)}\}_{0 \leq j \leq r-1}$ on the fibers F_{t_i}. Blow up all of these points to obtain the birational morphism $\pi : \overline{L} \longrightarrow L$. Then denoting F_i' the proper transforms of fibers L_{t_i}, we set $\tilde{L} = \overline{L} \setminus \cup_{i=1}^n F_i'$. We have a natural fibration $\tilde{L} \longrightarrow C$. We can construct the following diagram where π_2 is given by the apparent singular points $q_1 + \cdots + q_N$ and Ψ is given by the points $(q_1, p_1) + \cdots + (q_N, p_N)$. By an easy argument, for generic $\boldsymbol{\nu}$, we can also define a rational map $\tilde{\Psi} : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d) \cdots \to \mathrm{Hilb}^N(\tilde{L})$, where $\mathrm{Hilb}^N(\tilde{L})$ is the Hilbert scheme of N-points on the open surface $\tilde{L}$. Note that $\tilde{L}$ has a holomorphic symplectic structure, so $\mathrm{Hilb}^N(\tilde{L})$ becomes an (open) symplectic manifold.

$$\begin{array}{ccc}
\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d) & \overset{\tilde{\Psi}}{\cdots\to} & \mathrm{Hilb}^N(\tilde{L}) \\
\cup & & \downarrow \\
\mathcal{M}^{00} & \xrightarrow{\Psi} & S^N(\tilde{L}) = \tilde{L}^N/\mathfrak{S}_N \\
\pi_2 \downarrow & & \downarrow \\
S^N(C) & = & S^N(C) = C^N/\mathfrak{S}_N
\end{array}$$

We can prove the following proposition (cf. [15]).

Proposition 4.2. *The map Ψ is a birational morphism which preserves the holomorphic symplectic structures. Moreover the map $\pi_2 : \mathcal{M}^{00} \longrightarrow S^N(C)$ obtained by apparent singular points gives a Lagrangian fibration.*

We expect that $\mathcal{M}^{00}$ is Zarsiki dense in $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$. If this is true, the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ is birational to the Hilbert scheme $\mathrm{Hilb}^N(\tilde{L})$, hence irreducible.

The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ becomes a phase space of the differential equations of isomonodromic deformations and its deformation becomes a phase space of Hitchin integrable systems. Proposition 4.2 explain the reason why the method of separation of variables can be applicable to many integrable systems. We expect that more detailed birational geometry of the moduli space of connections and Hilbert schemes may be possible.

5. Two Lagrangian fibrations on the moduli spaces of connections.

5.1. **General cases.** From the argument above, assuming the genericity of local exponents $\boldsymbol{\nu}$, we have n two birational fibrations π_1, π_2 on the Zariski open set $\mathcal{M}^{00}$ of $\mathcal{M}^{\alpha}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ in Section 4, which make the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}^{00} & \xrightarrow{\pi_1} & \mathcal{P}^r_d \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ S^N(C) & \xrightarrow{p_2} & Jac(C) \simeq \mathrm{Pic}^d(C) \end{array} \tag{18}$$

As we see in the previous two sections, the morphism π_1, π_2 are both Lagrangian fibrations. Two maps p_1, p_2 can be given by $p_1((E,l)) = \wedge^r(E)$ and $p_2(q_1 + \cdots + q_N) = L^{-\frac{r(r-1)}{2}} \otimes \mathcal{O}_C(q_1 + \cdots + q_N)$.

In [11], [12], we investigate the relation of two Lagrangian fibrations.

5.2. **Painlevé VI case.** In the case of $C = \mathbf{P}^1$, $r = 2$, $d = -1$ $n = 4$, corresponding to the phase space of Painlevé VI equations, the moduli space of $\boldsymbol{\nu}$-parabolic connections $M_{\boldsymbol{\nu}}$ is constructed in [6], [7] as well as its natural compactification $\overline{M}_{\boldsymbol{\nu}}$.

If we assume that $\boldsymbol{\nu}$ is generic, $\overline{M_{\boldsymbol{\nu}}}$ is isomorphic to a smooth projective surface $S_{\boldsymbol{\nu}}$ obtained by the blowing up 8 points $\{b_i^{\pm} \in F_i\}_{1 \leq i \leq 4}$ of the Hirzebruch surface Σ_2 of degree 2. Moreover its anticanonical divisor is given by $-K_{S_{\boldsymbol{\nu}}} = Y = 2C_0 + F_1' + F_2' + F_3' + F_4'$ and $M_{\boldsymbol{\nu}} = S_{\boldsymbol{\nu}} \setminus Y$. The pair $(S_{\boldsymbol{\nu}}, Y)$ is called an Okamoto-Painlevé pair of type $D_4^{(1)}$ ([19], [16]). Here C_0 is the minimal section with $C_0^2 = -2$ and positive section $C_1 \sim C_0 + 2F$. In this case, the coarse moduli space $\mathcal{P}^1_{-1}$ of quasi-parabolic bundles $(E, l) = (\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), l)$ is a non-separated scheme obtained by $\mathbf{P}^1$ with 4 non-reduced double points at t_i. For simplicity, we forget non-reduced structure and identify $\mathcal{P}^1_{-1}$ with $\mathbf{P}^1$. The map π_1, π_2 can be extended to the compactification, and obtain the diagram (cf. [Section 9, [11]])

$$\begin{array}{ccc} \overline{M_{\boldsymbol{\nu}}} & \xrightarrow{\pi_1} & \mathbf{P}^1 \ni Q \\ {\scriptstyle \pi_2}\downarrow & & \\ \mathbf{P}^1 \ni q & & \end{array} \tag{19}$$

Proposition 5.1. *The morphism π_2 obtained by apparent singularity map is induced by the linear system $|F|$ of fibers of ruled surface $\Sigma_2 \longrightarrow \mathbf{P}^1$ and the forgetful morphism π_1 is obtained by the linear system $|L_1|$ with $L_1 \sim C_1 + F - E_1^+ - E_2^+ - E_3^+ - E_4^+$. (See Figure 1, 2).*

In Figure 1, 2, the coordinate q is the apparent singular point of the connection and Q is the coordinate of the quasi-parabolic structure on $E = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$.

Remark 5.1. By an explicit calculation, we see that the map $q \mapsto Q$ induces the birational map on $M_{\boldsymbol{\nu}}$ which is the so-called Okamoto mysterious Bäcklund transformation. Actually, we see that $\pi_2 : \overline{M}_{\boldsymbol{\nu}} \longrightarrow \mathbf{P}^1$ can be identified with the apparent singularity map $\overline{M}_{\boldsymbol{\nu}'} \longrightarrow \mathbf{P}^1$ with different local exponents $\boldsymbol{\nu}'$. In [12], we try to understand these two fibrations for the moduli space of $\boldsymbol{\nu}$-parabolic connections on $\mathbf{P}^1$ of rank 2 with $n \geq 5$, which corresponds to the phase spaces of Garnier systems ([14], [6]). On a Zariski open set of the moduli space, we see that two fibrations are really dual to each other coming from the product of projective space and its dual $\mathbf{P}^{n-3} \times (\mathbf{P}^{n-3})^{\vee}$ (cf. [12]).

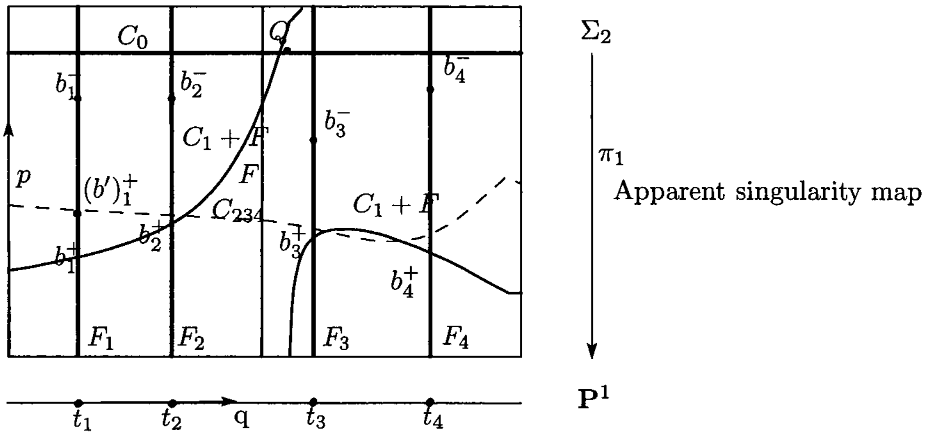
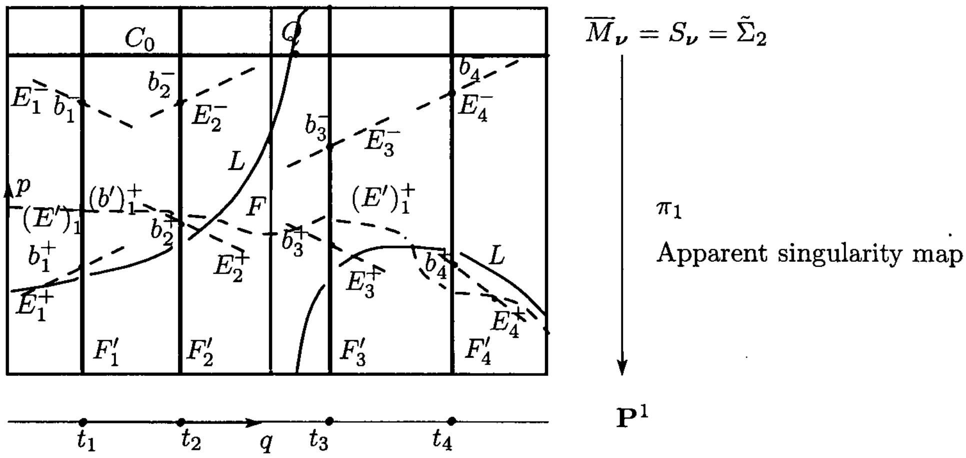


FIGURE 1. Apparent singularity map before blowing ups



$-K_S \sim Y = 2C_0 + F_1' + F_2' + F_3' + F_4'$

$S_\nu \simeq \overline{M_\nu}, \quad S_\nu \setminus Y \simeq M_\nu$

$L_1 \sim C_1 + F - E_1^+ - E_2^+ - E_3^+ - E_4^+$

$F \cdot L_1 = 1$

FIGURE 2. The Moduli space M_ν and the compactification $\overline{M_\nu}$. Blow-up 8 points of Hirzebruch surface Σ_2

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